Homework

November 4, 2019

1 Lecture 3

1. A function f is called uniformly convex of degree $\rho \geq 2$ on a set Q if there exists a constant $\sigma_\rho \geq 0$ such that, for all $x, y \in Q$ and $\nabla f(x) \in \partial f(x)$,

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\rho}{2} \|x - y\|^\rho.$$

Subgradient of f is said to be Hölder-continuous if for some $\nu \in [0, 1]$ and $L_\nu > 0$

$$\|\nabla f(x) - \nabla f(y)\| \leq L_\nu \|x - y\|^\nu.$$

Show that if f is uniformly convex then its conjugate f^* has Hölder-continuous subgradient and find the parameters ν, L_ν.

2. Find the Legendre–Fenchel conjugate for

$$\sum_{i=1}^{n} x_i \ln x_i, \quad \sum_{i=1} x_i = 1, \quad x_i \geq 0, i = 1, ..., n.$$

Note that by continuity $x \log x$ is defined to be 0 at $x = 0$.

3. Assume that the function f is Lipschitz-continuous with the constant M

$$|f(x) - f(y)| \leq M \|x - y\|_2.$$

Prove that $\mathrm{dom} f^*$ is bounded and find an estimate for the radius of the ball which contains $\mathrm{dom} f^*$.

4. Using the Legendre–Fenchel conjugate, show that, for all x, y

$$x^T y \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|_*^2,$$

where the conjugate norm $\|y\|_*$ is defined as

$$\|y\|_* = \max_x \{y^T x : \|x\| \leq 1\}.$$

In particular, for $p \geq 1$, $\|y\|_q$ is conjugate for $\|x\|_p$, where $1/p + 1/q = 1$ with the convention that $1/1 + 1/\infty = 1$.